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Transient dynamics of linear quantum amplifiers

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Abstract. The transient dynamics of a quantum linear amplifier during the transition from damping to amplification regime is studied. The master equation for the quantized mode of the field is solved, and the solution is used to describe the statistics of the output field. The conditions under which a nonclassical input field may retain nonclassical features at the output of the amplifier are analyzed and compared to the results of earlier theories. As an application we give a dynamical description of the departure of the system from thermal equilibrium.

PACS. 42.50.Ar Photon statistics and coherence theory – 42.50.Dv Nonclassical states of the electromagnetic field, including entangled photon states; quantum state engineering and measurements – 42.50.Lc Quantum fluctuations, quantum noise, and quantum jumps

Highlight Paper

1 Introduction

The master equation describing linear amplification or gain has been a prototype for discussing many questions in Quantum Optics. It was derived by the elimination of an unobserved environment using what has been termed a Born-Markov approximation [1]. Thus its properties were mainly determined by physical considerations, but it arrived at a form later to be shown to be the consistent generator of dissipative time evolution in quantum theory; thus it is of the Lindblad form [2].

The master equation for linear amplification was earlier represented as the generic model for an optical amplifier or attenuator [3]. It also describes the onset of laser oscillations until the time when nonlinear saturation starts to affect the behavior. In the trapped ion context, when the ion trap potential is regarded in a harmonic approximation, the cooling by lasers may be considered as an attenuation described by the same equation [4].

Its advantage is that it is exactly solvable, which allows us to follow the onset of gain or the damped approach to a steady state. The exact solution also allows us to evaluate the noise properties exactly and investigate the fading of nonclassical features of the initial state.

In all applications so far, the amplifying and attenuating coefficients of the equation have been regarded as constants. This corresponds to the assumption that the population inversion is instantaneously reached and the evolution starts from an initial state experiencing no previous evolution. Such theory, however, does not describe the transient dynamics of the linear amplifier, i.e. the dynamics when the pumping field is switched on or off. In this paper we generalize the previous theory of linear amplifiers to the case in which a smooth onset of amplification or attenuation takes place, and hence the amplifying and attenuation coefficients are time dependent. We solve the master equation with time dependent coefficients in terms of the characteristic function [5,6] and we use the solution to describe the transient dynamics of the linear quantum amplifier, one of the most widely used and common devices in quantum optics.

We choose to consider a situation where the gain medium is switched smoothly from an attenuating regime to an amplifying one. This takes place within a time interval centered at some definite time, before which we have a damped situation. Thus the initial gain, which is normalized to unity of course, decreases first until the amplification coefficient changes sign and gain starts to grow. The model allows us to follow the solution through this point, and we can see how the time dependence affects the noise properties and the possibility to retain initially imposed nonclassical features of the system state. As we may expect, the situation is more complicated than the simple constant coefficient case. It is, however, possible to retain earlier results on amplifier added noise and quantum cloning limits by considering the appropriate limiting cases.

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Theoretical works on optical transients of physical phenomena which cause amplification of light have received in the past a huge deal of attention [7]. To the best of the authors' knowledge, however, this is the first analytic description of the transient of phase-insensitive quantum linear amplifiers. Therefore the results presented in this paper give a clear contribution to the fundamental research in the theory of lasers and optical amplifiers since every linear amplifier or laser undergoes a transient behavior before stabilizing.

Our results can be directly applied to describe transients in technological applications based on linear amplifiers. Optical linear amplifiers are essential components of state-of-the-art optical networks. In order to attain best performances of the networks, however, it is crucial to analyze the behavior of linear amplifiers during power transients causing fast switching on and off of the amplifiers [8]. Although the linear amplifiers currently used in optical networks do not need to operate at the quantum level, the recent development of quantum technologies such as quantum communication, quantum cryptography and quantum computation, is based on the implementation of networks containing nanodevices and nanocomponents operating at the quantum level.

Recently a scheme for optimal cloning of coherent states with phase-insensitive linear amplifiers and beams splitters has been proposed [9]. Recent advances in the field of nanoelectromechanical systems have paved the way to the realization of experiments close to achieve the quantum limited detection and amplification. In [10], position resolution very close to the quantum limit is obtained, demonstrating the near-ideal performance of a single-electron transistor as a linear amplifier. Another recent application of linear amplifiers consists in a method for reconstructing a multimode entangled state [11]. Also quasiprobability functions have been shown to be measurable via direct photodetection of a linear amplified field [12].

The paper is structured as follows. Section 2 summarizes the results of earlier investigations for easy comparison with the present results. Section 3 presents the solution for time dependent coefficients and discusses its main properties. In Section 4 we discuss the possibilities to retain nonclassical features in the output of the amplifier relating our results to earlier calculations. In Section 5 we discuss the emergence of thermal features in the solution. Finally Section 6 concludes the discussion of the work.

2 Review on phase insensitive narrow band linear amplifiers

The simplest standard amplifier configuration consists of an assembly of N two-level atoms, of which N_2 are excited and N_1 are unexcited, interacting with a single-mode quantum field. It is assumed that the field frequency is resonant with the atomic frequency and that the population of the two-level atoms is partly inverted, i.e. $N_2 > N_1$. This is the standard model of a laser; its linear operation regime describes an amplifier [1]. In the standard description of linear amplifiers it is assumed that N_2 and N_1 are maintained approximately constant in time by some pump and loss mechanism.

Starting from a microscopic description of the interaction between the two-level atoms and the quantum mode, it is possible to derive the following master equation for the field mode in the interaction picture [1]

$$\frac{d\rho}{dt} = -\frac{C}{2}\left[a^\dagger a\rho - 2a\rho a^\dagger + \rho a^\dagger a\right] - \frac{A}{2}\left[aa^\dagger\rho - 2a^\dagger\rho a + \rho aa^\dagger\right], \quad (1)$$

with a and $a^\dagger$ annihilation and creation operator of the quantum harmonic oscillator and

$$A = \frac{2g^2}{\gamma^2}r_2, \quad (2)$$

$$C = \frac{2g^2}{\gamma^2}r_1, \quad (3)$$

where g is the coupling strength of the interaction between the two-level atoms and the mode of the field, $r_i = N_i/\gamma$ (with $i = 1, 2$) is the pumping rate into the atomic level i, and γ is a rate of the same order of the atomic linewidth.

A relevant quantity in the dynamics is the linear gain (or damping) factor, describing the linear growth (loss) of energy in the mode,

$$W = A - C. \quad (4)$$

When $W > 0$, the master equation describes a linear amplifier, when $W < 0$ it describes a linear absorber. The constant A gives the noise provided by the spontaneous emission; this term is present even if the mode energy is initially zero.

The time evolution of the amplitude of the field is described by the equation:

$$\langle a\rangle_{out} \equiv \langle a(t)\rangle = G^{1/2}e^{-i\omega_0 t}\langle a\rangle_{in}, \quad (5)$$

where $\langle a\rangle_{in} = \langle a(t=0)\rangle$, ω_0 is the frequency of the radiation field and the gain G is defined as

$$G = e^{Wt}. \quad (6)$$

The gain is greater than 1 for linear amplifiers and smaller than 1 for linear attenuators. The solution of the Fokker-Plank equation for the Glauber-Sudarshan P representation of the density matrix (P function) can be used to calculate the transformation of any incoming $P_{in}(\alpha)$ function by the amplifier:

$$P_{\rm out}(\alpha) = \int d\alpha_0 P(\alpha, t|\alpha_0)P_{in}(\alpha_0) \quad (7)$$

where

$$P(\alpha, t|\alpha_0) = \frac{1}{\pi m(t)}\exp\left[-\frac{\left|\alpha - G^{1/2}e^{-i\omega_0 t}\alpha_0\right|^2}{m(t)}\right] \quad (8)$$

is the amplifier transfer function [13]. The time dependent width is given by

$$m(t) = A\left[G(t) - 1\right]/W. \quad (9)$$

This quantity represents the average photon number of the spontaneous emission field [1]. Note that, for a linear amplifier, the gain grows asymptotically to infinity for $t \to \infty$, and so does the width [3]. For an absorber, on the other hand, the asymptotic value of the width $m(t)$ is finite.

By using equations (7) and (8) one can calculate the noise of the output field, defined as the symmetrically ordered fluctuations of the field mode [14]:

$$\begin{aligned} |\Delta a|^2_{out} &= \frac{1}{2}\langle a^\dagger a + a a^\dagger\rangle_{out} - \langle a\rangle_{out}\langle a^\dagger\rangle_{out} \\ &= G|\Delta a|^2_{in} + m(t) - \frac{1}{2}(G-1) \\ &\equiv G\left(|\Delta a|^2_{in} + \mathcal{A}\right), \end{aligned} \quad (10)$$

where

$$\mathcal{A} = \frac{1}{2}\left(\frac{A+C}{A-C}\right)\left(1 - \frac{1}{G}\right), \quad (11)$$

is the equivalent noise factor, or amplifier added noise, introduced by Caves [14]. This quantity describes the fluctuations of the internal modes of the amplifying medium. Since the input field and the internal modes of the amplifying medium are uncorrelated, their fluctuations add in quadrature and they are both amplified. The minimum value of the added noise for infinite gain, e.g. for $t \to \infty$, is given by the Caves limit:

$$\mathcal{A}_C = \frac{1}{2}\left(\frac{A+C}{A-C}\right) = \frac{1}{2} + \theta \geq \frac{1}{2}, \quad (12)$$

where the excess noise factor θ gives the initial mean number of excitations of the internal modes of the medium, and therefore approaches zero when the initial temperature of the amplifying medium vanishes, $T \to 0$.

It has been demonstrated that the output field of a phase insensitive narrow band linear amplifier may possess nonclassical features only if the input field is nonclassical. However, as the amplifier gain increases, any nonclassical feature of the light, which was present in the input field, tends to be lost. In particular, subPoissonian statistics and squeezing are lost when the gain G exceeds the value 2 [1,13].

In the next section we present a theory describing the transient regime of the amplification process. In other words, we will drop the assumption that N_2 and N_1 are constant and we will describe the onset of the amplification process from an initial damping regime. Our aim is to study the transient dynamics and to investigate how the results for the standard amplifier, described in this section, are modified.

3 Transient regime of linear amplification

Previous work on linear amplifiers deals with a situation in which the amplifying medium, e.g. an assembly of two-level atoms, satisfies the population inversion condition required to amplify an input field. In order to reach the inverted population condition it is necessary to pump the atoms from their initial thermal condition till the point in which $N_2 > N_1$. During this transient regime the pumping rates to levels 2 and 1 ($r_2(t)$ and $r_1(t)$) change with time till they reach a stationary value for which $N_2/N_1 \propto r_2/r_1 > 1$ (amplification regime). In this case, the time evolution of the field mode is described by a master equation of the same form of equation (1), but with time dependent coefficients $A(t)$ and $C(t)$. Similarly, if one switches off the external pump, the ratio of the atomic populations N_2/N_1 will tend to the Boltzmann factor and the amplification process will eventually stop, the system approaching its thermal equilibrium. In the following we consider the first of these two situations, i.e. the onset of amplification due to the creation of population inversion in an initially damping medium. In more detail, we consider the case in which

$$A(t) = \frac{2g^2}{\gamma^2}r_2(t) = A\frac{e^{\varepsilon(t-t_0)}}{e^{\varepsilon(t-t_0)} + e^{-\varepsilon(t-t_0)}} + B, \quad (13)$$

$$C(t) = \frac{2g^2}{\gamma^2}r_1(t) = A\frac{e^{-\varepsilon(t-t_0)}}{e^{\varepsilon(t-t_0)} + e^{-\varepsilon(t-t_0)}} + B, \quad (14)$$

where ε is the rate of change of the pumping coefficients, that is the amplification onset rate. We assume that at $t = -\infty$ the state of the ensemble of two-level atoms constituting the amplifying medium is thermal, that is

$$\begin{aligned} \frac{N_2(t \to -\infty)}{N_1(t \to -\infty)} &= \frac{A(t \to -\infty)}{C(t \to -\infty)} = \frac{B}{A+B} \\ &= e^{-\hbar\omega_0/k_BT}, \end{aligned} \quad (15)$$

which implies $B/A = (e^{\hbar\omega_0/k_BT} - 1)^{-1} = n_M$, with n_M mean number of excitations of the medium. We assume that the state of the amplifying medium practically does not change in the time interval $-\infty < t \leq 0$.

Under these conditions the asymptotic gain factor $W(t)$ takes the form

$$W(t) = A(t) - C(t) = A\tanh[\varepsilon(t - t_0)]. \quad (16)$$

Note that, for $t < t_0$, $W(t) < 0$ and the system behaves as an absorber, while for $t > t_0$, $W(t) > 0$ and the system behaves as an amplifier. Therefore t_0 indicates the time at which the amplification process begins. From equation (16) we infer that the constant A is the asymptotic gain factor. Finally we stress that, for $t \to \infty$,

$$\begin{aligned} &N_2(t\to\infty)/N_1(t\to\infty) = A(t\to\infty)/C(t\to\infty) \\ &\quad = (A+B)/B = N_1(t\to-\infty)/N_2(t\to-\infty), \end{aligned} \quad (17)$$

i.e., the population of the exited (ground) state tends asymptotically to the initial population of the ground (excited) state.

3.1 The master equation and its solution

The master equation describing the transient behavior of the amplification process, in the interaction picture, is the following

$$\frac{d\rho}{d\tau} = -\frac{A'(\tau)}{2}\left[aa^\dagger\rho - 2a^\dagger\rho a + \rho a a^\dagger\right] - \frac{C'(\tau)}{2}\left[a^\dagger a\rho - 2a\rho a^\dagger + \rho a^\dagger a\right], \quad (18)$$

with

$$A'(\tau) = A'\frac{e^{(\tau-\tau_0)}}{e^{(\tau-\tau_0)} + e^{-(\tau-\tau_0)}} + B', \quad (19)$$

$$C'(\tau) = A'\frac{e^{-(\tau-\tau_0)}}{e^{(\tau-\tau_0)} + e^{-(\tau-\tau_0)}} + B'. \quad (20)$$

In the previous equations we have introduced the relevant physical dimensionless parameters $\tau = \varepsilon t$, $\tau_0 = \varepsilon t_0$, $A' = A/\varepsilon$ and $B' = B/\varepsilon$. As we will see in the following, the parameter A', which is the ratio between the asymptotic gain factor and the rate of onset of the amplification, plays a central role in the system dynamics. Indeed both the gain $G(\tau)$ and the added noise depend crucially on this parameter.

Following the method developed in [5] we solve the master equation given by equation (18) in terms of the quantum characteristic function (QCF) [15], defined through the equation

$$\rho_S(\tau) = \frac{1}{2\pi}\int \chi_\tau(\xi)\, e^{\left(\xi^* a - \xi a^\dagger\right)} d^2\xi. \quad (21)$$

The solution reads as follows

$$\chi_\tau(\xi) = e^{-\Delta(\tau)|\xi|^2}\chi_0\left(G^{1/2}(\tau)e^{-i(\omega_0/\varepsilon)\tau}\xi\right), \quad (22)$$

where χ_0 is the QCF of the initial state of the field, ω_0 is the field frequency and the $G(\tau)$ is the gain, given by

$$G(\tau) = e^{\int_0^\tau W(\tau')d\tau'}. \quad (23)$$

The quantity $\Delta(\tau)$, appearing in equation (22) is defined as follows

$$\Delta(\tau) = \frac{1}{2}[G(\tau)]\int_0^t [G(\tau')]^{-1}\left[C'(\tau') + A'(\tau')\right] d\tau'. \quad (24)$$

It is worth underlining that the solution given by equation (22), with the help of equations (23, 24), holds whatever the explicit time dependence of the coefficients $A(\tau)$ and $C(\tau)$, appearing in equation (18), is. The case considered in the paper [Eqs. (13, 14)] has been chosen to illustrate the transient dynamics in a physically reasonable and well justified model. Indeed equations (13, 14) describe a situation in which from an initial condition in which $N_1 > N_2$, the populations of the two-level systems constituting the amplifying medium pass smoothly to the inversion condition $N_2 > N_1$ necessary for amplification. In passing, we note that the hyperbolic tangent time dependence is one of the most commonly adopted phenomenological models in the description of transients of physical systems.

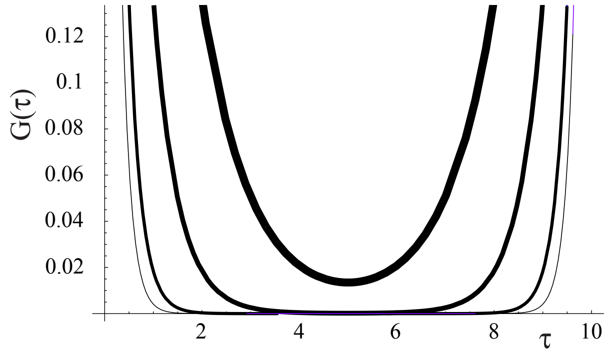

Fig. 1. Time evolution of $G(\tau)$ for $A' = 2, 3, 4, 5.5$ (increasing values of A' correspond to decreasing thickness of the line) and $\tau_0 = 5$.

Starting from equation (22), one can calculate the Wigner function, the Glauber-Sudarshan P function, and the Husimi Q function by means of the relation [15]

$$W_\tau(\alpha, p) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} d^2\xi\chi_\tau(\xi)\exp(\alpha\xi^* - \alpha^*\xi)e^{(p|\xi|^2/2)}, \quad (25)$$

where $p = -1, 0, 1$ corresponds to the Q, Wigner, and P functions, respectively. In particular, carrying out the calculations, it turns out that the P function has the same form of equation (8), but with $G(\tau)$ given by equation (23) and $m(\tau) = [G(\tau) - 1]/2 + \Delta(\tau)$.

Inserting equation (16) into equation (23) and carrying out the integration yields

$$G(\tau) = \left[\frac{\cosh(\tau - \tau_0)}{\cosh(\tau_0)}\right]^{A'}. \quad (26)$$

It is not difficult to prove that, for $\tau_0 = 0$, and in the limit of infinitely fast onset of the amplification ($\varepsilon \to \infty$), the gain function tends to $G(t) = e^{At} = e^{Wt}$ [see Eq. (6)].

In Figure 1 we plot the gain $G(\tau)$ for four increasing values of A'. As clearly shown in the figure for increasing values of A' the values of the gain in proximity of the amplification time τ_0 become smaller and smaller. This is because small values of A' correspond to small values of the asymptotic gain factor or, equivalently, to a very slow amplification onset rate. In general, the gain decreases for times smaller than τ_0 and, as expected, starts to increase after the amplification sets in, even if $G(\tau) > 1$ only for times $\tau > 2\tau_0$. Note that the standard theory of linear amplification predicts that for a linear amplifier it is always $G(\tau) > 1$ (see Sect. 2). However, if one takes into account the transient regime characterizing the initial dynamics of every linear amplifier it turns out that there exist an interval of time at which, although $W > 0$, there is still no gain. The reason why the gain becomes greater than 1 only after the time $2\tau_0$ is that for $0 < \tau < \tau_0$ the system is in a damping regime, and hence the gain decreases. It takes exactly another interval of time τ_0 after the onset of the amplification process to undo the initial decrease in the gain. At $\tau = 2\tau_0$ we have $G(2\tau_0) = G(0) = 1$ after which the gain increases monotonically. This is a new feature brought to light by our theory.

Let us focus on the quantity $\Delta(\tau)$. Inserting equation (26) into equation (24) we get

$$\Delta(\tau) = G(\tau)\frac{A'+2B'}{2}\int_0^\tau \left[\frac{\cosh(\tau'-\tau_0)}{\cosh(\tau_0)}\right]^{-A'} d\tau' \equiv G(\tau)\frac{A'+2B'}{2} I_{A'}(\tau-\tau_0). \quad (27)$$

For $\tau_0 = 0$ and for each A' real, with $A' \neq 1$, we have

$$I_{A'}(\tau) = \frac{\sinh\tau}{(A'-1)}\frac{1}{(\cosh\tau)^{A'-1}} \times F[1, 1-A'/2, 3/2-A'/2; (\cosh\tau)^2], \quad (28)$$

with $F[1, 1-A'/2, 3/2-A'/2; (\cosh\tau)^2]$ being the hypergeometric function of the variable $x = (\cosh\tau)^2$. For $A' = 1$ the integral appearing in equation (27) is simply equal to

$$I_1(\tau) = 2\arctan(e^\tau). \quad (29)$$

The mathematical expression of the added noise in the special case of integer values of A' is discussed in Appendix A. For $\tau_0 \neq 0$, one gets

$$\Delta(\tau) = G(\tau)\frac{A'+2B'}{2}\cosh(\tau_0)^{A'} \times \left[I_{A'}(\tau-\tau_0) + I_{A'}(\tau_0)\right], \quad (30)$$

where $I_{A'}(\tau-\tau_0)$ and $I_{A'}(\tau_0)$ are obtained from equation (28) by substituting for the variable τ the expressions $\tau-\tau_0$ and τ_0, respectively.

3.2 Noise of the output field

From the QCF solution given by equation (22) we can easily calculate the mean values of observables of interest, e.g. those characterizing the output field statistics, by means of the relation [15]

$$\langle a^{\dagger m} a^n \rangle = \left(\frac{d}{d\xi}\right)^m \left(-\frac{d}{d\xi^*}\right)^n e^{|\xi|^2/2}\chi(\xi)\bigg|_{\xi=0}. \quad (31)$$

We look first of all at the symmetrically ordered fluctuation, as defined by Caves [14]

$$|\Delta a|^2_{out} = G(\tau)|\Delta a|^2_{in} + \Delta(\tau) = G(\tau)\left[|\Delta a|^2_{in} + \mathcal{A}(\tau)\right], \quad (32)$$

The added noise is given by

$$\mathcal{A} = \Delta(\tau)/G(\tau), \quad (33)$$

with $\Delta(\tau)$ given by equation (30). This quantity is clearly different from the one given by equation (11) for the standard linear amplifier case. It is possible to show that for infinite gain, e.g. for $\tau \to \infty$, this quantity is always greater or equal to $\mathcal{A}_C$, $\mathcal{A}_C$ being the Caves limit for an infinitely fast onset of the amplification process.

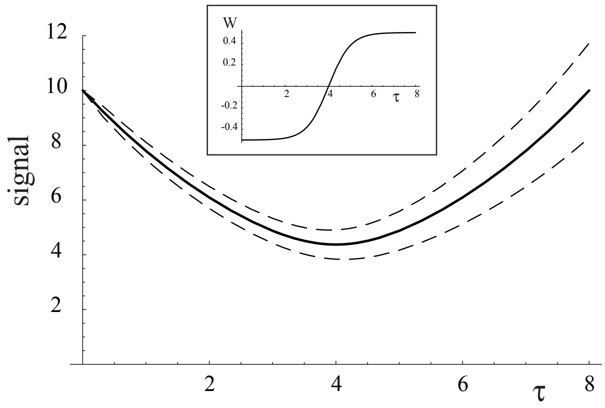

Fig. 2. Damping and amplification of an initial input field having $\langle a(t=0)\rangle = 10$, for $\tau_0 = 4$, $A' = 0.5$ and $B' = 0.5 \times 10^{-2}$. The dashed lines indicate signal width $\Delta(\tau)^{1/2}$. The insert shows the gain factor $W(\tau) = A(\tau) - C(\tau)$ in the same time interval and for the same values of the parameters.

In order to derive the Caves limit from equation (33), we note that for $\tau \to \infty$, $I_{A'}$, as given by equation (28), tends to [16]

$$I^\infty_{A'} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(A'/2)}{\Gamma[(A'+1)/2]}. \quad (34)$$

Therefore the asymptotic value of the added noise is

$$\mathcal{A} = \frac{A'+2B'}{2} I_{A'} \to \frac{A'+2B'}{2}\frac{\sqrt{\pi}}{2}\frac{\Gamma(A'/2)}{\Gamma[(A'+1)/2]} = \left(\frac{1}{2}+\frac{B}{A}\right)\sqrt{\pi}\frac{\Gamma(A'/2+1)}{\Gamma[(A'+1)/2]}. \quad (35)$$

The Caves limit is obtained for an infinitely fast onset of the amplification process, that is for $\varepsilon \to \infty$, i.e. $A' \to 0$. Substituting $A' = 0$ into equation (35), and remembering that $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$, one gets

$$\mathcal{A}_C = \frac{1}{2} + \frac{B}{A} \geq \frac{1}{2}, \quad (36)$$

that is the Caves limit [see Eq. (12)]. From equation (35) one can see that, for fixed values of the initial mean number of excitations of the medium $n_M = B/A$, $\mathcal{A}_C$ is actually the smallest asymptotic value of the added noise.

A careful analysis of the noise at the output field, as given by equation (32), shows that this quantity, as one would expect, increases monotonically with time whatever the initial state is. In Figure 2 we show the dynamics of an input field which is damped for $\tau < \tau_0 = 4$ and then amplified for $\tau > \tau_0 = 4$. The figure shows that the width of the signal always increases. In the following section we will study in more detail the transient dynamics for different types of input fields. Since we are dealing with phase insensitive amplifiers/absorbers, it turns out that it is not possible to generate nonclassical states from classical input fields. However, it is possible to analyze how the conditions to retain initial nonclassical features are modified due to the transient dynamics. In addition we will look at the field statistics by explicitly calculating the time evolution of the Wigner function.

4 Nonclassical properties of the output field

4.1 Squeezing and subPoissonian statistics

Let us begin studying the conditions for which the output field can retain squeezing properties when the input state is a squeezed state of the electromagnetic field. We define the dimensionless quadratures of the field as follows

$$u = \frac{1}{\sqrt{2}}\left(a + a^\dagger\right), \tag{37}$$

$$v = \frac{-i}{\sqrt{2}}\left(a - a^\dagger\right). \tag{38}$$

The squeezed states satisfy the minimum uncertainty relation $\Delta u \Delta v = 1/2$, but are characterized by an unequal distribution of the quantum fluctuations

$$\Delta u = \frac{s}{\sqrt{2}}, \qquad \Delta v = \frac{1}{\sqrt{2}s}, \tag{39}$$

with $s \neq 1$. Introducing the rotating coordinates

$$\tilde{u} = u\cos(\omega_0 t) - v\sin(\omega_0 t), \tag{40}$$

$$\tilde{v} = v\cos(\omega_0 t) + u\sin(\omega_0 t), \tag{41}$$

and using equation (31) we get

$$(\Delta\tilde{u})^2_{out} = G(\tau)\left[(\Delta\tilde{u})^2_{in} + \mathcal{A}\right], \tag{42}$$

$$(\Delta\tilde{v})^2_{out} = G(\tau)\left[(\Delta\tilde{v})^2_{in} + \mathcal{A}\right], \tag{43}$$

with $\mathcal{A}$ given by equations (33) and (27). For an input squeezed state having $s < 1$, the output can remain squeezed if and only if

$$G(\tau)\left[(\Delta\tilde{u})^2_{in} + \mathcal{A}\right] < \frac{1}{2}. \tag{44}$$

It is possible to show that the maximum allowed value of the gain $G(\tau)$, in order to retain squeezing at the output, decreases with A' and B'. In other words, for increasing values of A' and B', one can retain squeezing only for smaller and smaller values of the gain (less efficient amplification). Having in mind equation (30), one finds that, for $\tau_0 = 0$, the output field is still squeezed if the gain satisfies the following inequality

$$G(\tau) < \frac{1}{s^2 + (A' + 2B')I_{A'}(\tau)}. \tag{45}$$

It is worth recalling the standard result for phase insensitive linear amplifiers, which states that the upper limit for the gain compatible with squeezing at of the output field is $G = 2$, the magic number for photon cloning [13]. The analysis of the behavior of the gain in our case is more complicated, since the r.h.s. of the inequality (45) depends on time. A numerical study shows that, although for certain time intervals, the r.h.s of the inequality may be greater than 2, in these time intervals $G(\tau)$ is always smaller than 2. Hence, also in the case studied in this paper $G = 2$ constitutes an upper limit for retaining nonclassical features in the output field.

Let us now look at the dynamics of an input Fock state. We recall that one of the nonclassical features of such states is that their statistics is subPoissonian. Similarly to what we have done for the squeezed states we analyze the requirements to retain subPoissonian statistics at the output field. To this aim we introduce the Mandel parameter Q [1]

$$Q = \frac{\langle n^2\rangle - \langle n\rangle^2}{\langle n\rangle} - 1. \tag{46}$$

This quantity gives an indication of the statistics of a quantized field. For a Fock state Q takes its lowest value $Q = -1$ while for a coherent state Q is equal to 0. Therefore, values of $Q < 0$ indicate subPoissonian statistics, while $Q = 0$ indicates Poissonian statistics and $Q > 0$ superPoissonian statistics. Using equation (31) we derive the time evolution of the Mandel parameter as follows

$$Q_{out} = \frac{\langle n\rangle^2_{out} + [G(\tau)]^2\langle n\rangle_{in}[Q_{in} - \langle n\rangle_{in}]}{\langle n\rangle_{out}}, \tag{47}$$

where $\langle n\rangle_{out}$ and $\langle n\rangle_{in}$ are the mean number of photons of the output and input fields, respectively, and

$$\begin{aligned}\langle n\rangle_{out} &= G(\tau)\langle n\rangle_{in} + \frac{1}{2}[G(\tau) - 1] + \Delta(\tau)\\ &= G(\tau)\langle n\rangle_{in} + m(\tau).\end{aligned} \tag{48}$$

For a Fock input state $Q_{in} = -1$ and $\langle n\rangle_{in} = n_0$, hence the condition for having subPoissonian statistics at the output is

$$\langle n\rangle_{out}(\langle n\rangle_{out} + 1) < G^2(\tau)n_0(n_0 + 1). \tag{49}$$

A numerical analysis shows that, as for the squeezing, also for the subPoissonian statistics, we obtain the limit 2 for the gain typical of the standard theory of linear amplifiers.

As an example, in Figure 3 we compare the Mandel parameters of the output fields for an initial Fock state $|n_0 = 5\rangle$ in the cases $A' = 0.05$ (fast onset of the amplification and/or small value of the asymptotic gain) and $A' = 1$. From the figure one sees that, for $A' = 0.05$, one can retain subPoissonian statistics of the output field for higher values of the gain compared to the $A' = 1$ case.

4.2 Wigner function

Let us now have a look at the complete statistic of the output field, by means of the Wigner function. Inserting equation (22) into equation (25), and putting $p = 0$ we get

$$W_\tau(\alpha) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} d^2\xi\, e^{-\Delta(\tau)|\xi|^2}\, e^{\alpha\xi^* - \alpha^*\xi} \times \chi_0(G^{1/2}(\tau)e^{-i\omega_0\tau/\varepsilon}\xi). \tag{50}$$

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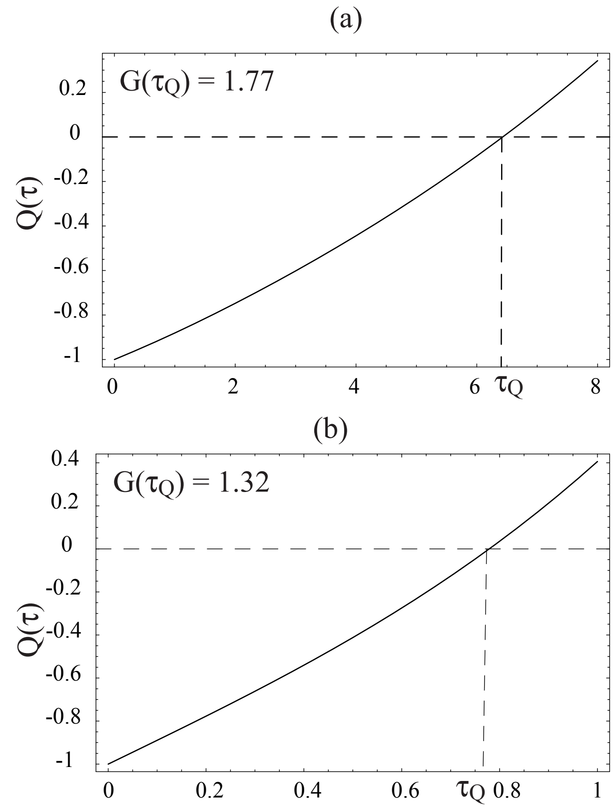

Fig. 3. Mandel parameters of the output fields for an initial Fock state $|n_0 = 5\rangle$ in the cases $A' = 0.05$ (a), and $A' = 1$ (b). We have set $\tau_0 = 0$ and $B/A = 10^{-2}$ in both (a) and (b). We indicate with τ_Q the instant of time at which $Q = 0$. In the upper-left corner we indicate the corresponding value of the gain at $\tau = \tau_Q$: $G(\tau_Q)$.

Inserting the inverse Fourier transform of equation (25) into equation (50) gives

$$\begin{aligned} W_\tau(\alpha) &= \frac{1}{\pi^2}\int_{-\infty}^{\infty} d^2\alpha_0 W_0(\alpha_0) \\ &\quad \times \int_{-\infty}^{\infty} d^2\xi\, e^{-\Delta(\tau)|\xi|^2}\, e^{b(\tau,\alpha,\alpha_0)\xi^* - b^*(\tau,\alpha,\alpha_0)\xi} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} d^2\alpha_0 W_0(\alpha_0) \frac{\exp\left[-\frac{|b(\tau,\alpha,\alpha_0)|^2}{\Delta(\tau)}\right]}{\Delta(\tau)} \\ &\equiv \frac{1}{\pi}\int_{-\infty}^{\infty} d^2\alpha_0 W_\tau(\alpha|\alpha_0) W_0(\alpha_0), \end{aligned} \tag{51}$$

with

$$b(\tau,\alpha,\alpha_0) = \alpha - \alpha_0 G^{1/2}(\tau) e^{i\omega_0\tau/\varepsilon}. \tag{52}$$

In the derivation of equation (51) we have used the property that the Fourier transform of a Gaussian is a Gaussian. The quantity $W_\tau(\alpha|\alpha_0)$ is the propagator which, for $\tau \to 0$, tends to the delta function $\delta(\alpha - \alpha_0)$.

If the state of the input field is a coherent state $|\alpha_0\rangle$, than the Wigner function of the output state reads as follows

$$W_\tau(\alpha) = \frac{1}{\pi}\frac{\exp\left[-\frac{\left|\alpha_0 G^{1/2}(\tau)e^{i\omega_0\tau/\varepsilon} - \alpha\right|^2}{\Delta(\tau)+1/2}\right]}{\Delta(\tau)+1/2}. \tag{53}$$

The Wigner function of the output state is therefore a Gaussian. Having in mind the time evolution of $G(\tau)$ [see Eq. (26) and Fig. 1], one realizes that, in a frame rotating with the frequency ω_0, the Wigner function of an input coherent state $|\alpha_0\rangle$, with $\alpha_0 \neq 0$, moves towards the center of the phase space for $\tau < \tau_0$ and then moves away for $\tau > \tau_0$, while its width continuously increases.

We now consider the case of an initially squeezed state. The initial QCF for squeezed coherent state is

$$\chi_0(\xi) = \exp\left[-\frac{1}{2}|\xi C_r - \xi^* e^{-i\phi} S_r|^2 + i(\xi^*\alpha_0^* + \xi\alpha_0)\right]. \tag{54}$$

Here $C_r = \cosh(r)$ and $S_r = \sinh(r)$, α_0 is the displacement of the input field and $z = re^{-i\phi}$ is the squeezing argument.

For an input squeezed vacuum state ($\alpha_0 = 0$), with squeezing angle $\phi = 0$, the Wigner function at time τ takes the form

$$\begin{aligned} W_\tau(\alpha) &= \frac{1}{\pi^2}\int_{-\infty}^{\infty} d^2\xi\, e^{-\Delta(\tau)|\xi|^2} e^{(\alpha\xi^* - \alpha^*\xi)} \\ &\quad \times \exp\left[-\frac{1}{2}G(\tau)|e^{-i\omega_0\tau/\varepsilon}\xi C_r - e^{i\omega_0\tau/\varepsilon}\xi^* S_r|^2\right]. \end{aligned} \tag{55}$$

This Fourier transformation can be calculated with the method used in [17], the result being

$$\begin{aligned} W_\tau(\alpha) = M\Bigg\{&\exp\left[\frac{-2\alpha_x^2}{2\Delta(\tau) + G(\tau)(C_{2r} + S_{2r})^{-1}}\right] \\ &+ \exp\left[\frac{-2\alpha_y^2}{2\Delta(\tau)G(\tau)(C_{2r} - S_{2r})^{-1}}\right]\Bigg\}. \end{aligned} \tag{56}$$

Here, α_x and α_y are the real and imaginary parts of α, and M is a time dependent normalization constant. We note that this result is consistent with equations (42, 43) used in Section 4.1 to study the time evolution of the quadratures of the field for an initial input squeezed state. Contour plots showing the time evolution of the Wigner function are shown in Figure 4a, while in Figure 4b we show the time evolution of the squeezing of the quadrature amplitude $\tilde{u}$.

5 Departure from thermal equilibrium

The analytic approach we have described in the previous section to analyze the onset of the amplification process, can be used to study how a system departs from an initial thermal equilibrium situation. In more detail, we consider the case in which the medium, modelled as an ensemble of two-level atoms, is initially in thermal equilibrium

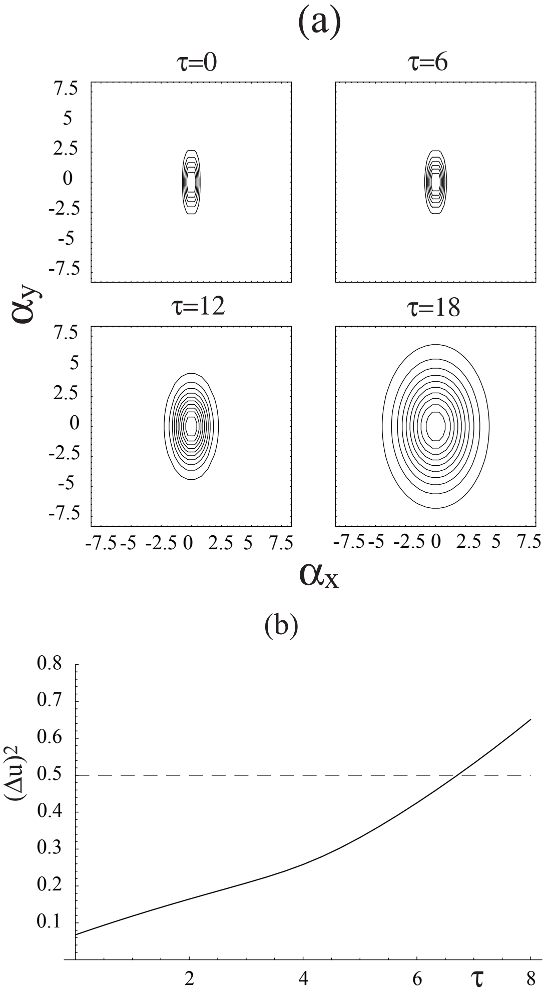

Fig. 4. Contour plots of the Wigner function at different times τ for an input squeezed state with $r = 1$. We have put $\tau_0 = 4$, $A' = 0.1$, $n_B = B/A = 0.1$ (a). Variance of the quadrature $\tilde{u}$ as a function of time, for the same values of the parameters (b).

with the mode of the quantized field. The ratio N_2/N_1 between the populations of the excited and ground states, respectively, is therefore given by the Boltzmann factor $N_2/N_1 = e^{-\hbar\omega_0/k_B T}$. The state of the field is a thermal state at T temperature.

At $\tau = 0$ one switches on pumping lasers which change the population of the two-level atoms until the condition of population inversion, necessary for the onset of the amplification process, is reached. The pumping lasers alter the initial condition of equilibrium between the medium and the system (the field mode). In order to study how the system departs from the condition of thermal equilibrium with the two-level atoms medium, we use the solution of the master equation (18) to calculate the time evolution of the Wigner function of the field. For an initial thermal state, the QCF at time τ has the form

$$\chi_\tau(\xi) = e^{-\Delta(\tau)|\xi|^2} \exp\left[-\left(\langle n\rangle_{out} + \frac{1}{2}\right) G(\tau)|\xi|^2\right]. \quad (57)$$

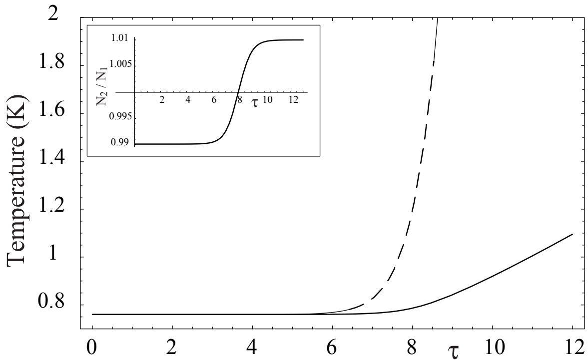

Fig. 5. Time evolution of the temperature of the system for $A' = 1$ (dotted line) and $A' = 0.05$ (solid line). For both graphics we have set $\omega_0 = 10^{14}$ Hz and $n_B = B/A = 10^3$. The box in the top left corner is the ratio between the populations of the two-level atoms. This quantity does not depend on A' and B' separately, but on the ratio $n_B = B/A$ only [see Eqs. (13) and (14)].

Inserting this equation into equation (25) one gets the following expression for the Wigner function at time τ

$$W_\tau(\alpha) = \frac{1}{\pi}\frac{1}{\langle n\rangle_{out} + 1/2} \exp\left[-\frac{|\alpha|^2}{\langle n\rangle_{out} + 1/2}\right]. \quad (58)$$

In the last two equations, $\langle n\rangle_{out}$ is the number of photons of the output field, as given by equation (48). The Wigner function of equation (58) is the Wigner function of a thermal state at a temperature $T(\tau)$ which varies with time. The medium is not in thermal equilibrium anymore, since the pumping lasers change the two-level atoms population until the population inversion condition is reached. However, equation (58) shows that the medium plus the pumping lasers behave, as far as the system (mode field) is concerned, as a thermal reservoir at varying temperature $T(\tau)$, as known from the theory of laser cooling [20]. Having in mind that $2\langle n\rangle_{out} + 1 = \coth[\hbar\omega_0/k_B T(\tau)]$ [1], and using the relation $\mathrm{arcoth}(x) = [\ln(x+1) - \ln(x-1)]/2$ (for $x^2 > 1$), we can express the time evolution of the temperature as follows:

$$T(\tau) = \frac{\hbar\omega_0}{k_B}\left\{\ln\left[\langle n\rangle_{out} + 1\right] - \ln\left[\langle n\rangle_{out}\right]\right\}^{-1}. \quad (59)$$

Figure 5 compares the behavior of $T(\tau)$ for the two cases $A' = 1$ and $A' = 0.05$. The figure shows clearly that in both cases the temperature of the system is constant during the damping regime and it starts to increase when approaching the population inversion at $\tau = \tau_0$, i.e. when the change in the population ratio N_2/N_1 becomes considerable. The increase in the temperature is much higher for higher values of A', since in this case the asymptotic gain is also higher.

In order to characterize further the departure from the initial condition of the system we look at the von-Neumann entropy of the field. We use the result obtained by Agarwal [19] to calculate the dynamical entropy for a

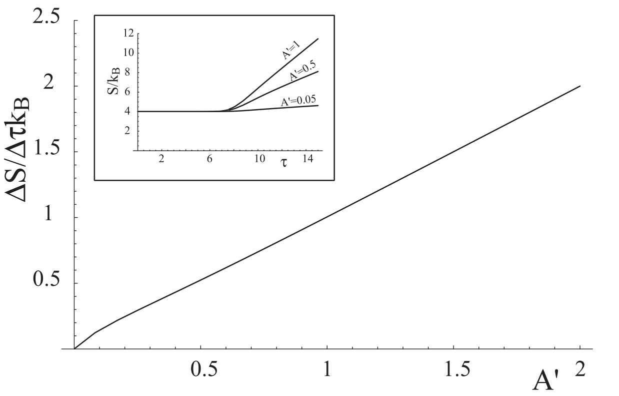

Fig. 6. Dependence of the entropy increase rate on A'. The entropy increase rate is defined as $\Delta S/\Delta\tau k_B = [S(\tau = 14) - S(\tau = 10)]/4k_B$. In the box in the top left corner the time evolution of the entropy for the three exemplary values $A' = 1$, $A' = 0.5$, and $A' = 0.05$ is shown. In all the plots we have set $\tau_0 = 8$, and $n_B = B/A = 10$.

state of the form given by equation (58)

$$S(\tau) = k_B \{[\langle n\rangle_{out} + 1] \ln[\langle n\rangle_{out} + 1] - \langle n\rangle_{out} \ln[\langle n\rangle_{out}]\}. \quad (60)$$

From direct inspection in the previous equation one sees that, similarly to the dynamics of the temperature, the entropy remains approximately constant for $\tau < \tau_0$ and begins to increase when $\tau \simeq \tau_0$. In Figure 6 we show how the entropy increase rate (in units of k_B) changes with A'; in the box in the top left corner the dynamics of the entropy for three example values of A' is shown. For increasing values of A', the linear increase in the entropy due to the amplification process is faster and faster. This result is in accordance with the behavior of the temperature of the system. In fact, smaller values of A' correspond to smaller asymptotic gain and therefore less efficient amplification processes.

6 Conclusions

In this paper we have discussed the dynamics of a quantum linear amplifier during the onset of the amplification process. For an amplifying medium consisting of an assembly of two-level atoms, our theory describes the dynamics of the output field when the medium passes from a condition in which the population of the atoms is thermal, to a condition of population inversion characterizing the amplifying regime.

We have solved exactly the master equation describing the transient dynamics of the linear amplifier in terms of the quantum characteristic function. The solution is used to investigate conditions under which an input nonclassical field may retain nonclassical features at the output of the linear amplifier. We derive the analytic expressions for the output noise, as well as for the squeezing, the Mandel parameter and the Wigner function of the output field, and we use them to characterize completely the transient dynamics of the output field.

Our results are compared with earlier theories of phase insensitive linear amplifiers which rely on the assumption that the population inversion is instantaneously reached, i.e. neglecting the transient regime. We show that also for a slow onset of amplification, the gain $G(\tau)$ has to be smaller than 2 (the cloning magic number) in order for the output field to retain initial nonclassical properties.

We conclude the paper analyzing the situation in which the initial mode of the field and the two-level atoms medium are in thermal equilibrium at T temperature. An external laser pumps up the atoms of the medium till the condition of population inversion is reached. This is an example of dynamic departure from a thermal equilibrium condition that can be studied analytically. We analyze the time evolution of the temperature and of the von-Neumann entropy on the characteristic parameters of the linear amplifier. We find that, as known from the theory of laser cooling, the medium plus the pumping lasers behave, as far as the system is concerned, as a thermal reservoir at varying temperature. We find that the entropy increase rate depends crucially on the asymptotic gain.

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Appendix A

For integer values of A' the integral $I_{A'}$ defined in equation (27) is given by [18]

$$I_{2m}(\tau) = \frac{\sinh\tau}{2m-1}\frac{1}{(\cosh\tau)^{2m-1}} \times \left[1 + \sum_{k=1}^{m-1} \frac{\Gamma(m)\Gamma(m-k-1/2)}{\Gamma(m-k)\Gamma(m-1/2)}(\cosh\tau)^{2k}\right], \quad (61)$$

$$I_{2m+1}(\tau) = \frac{\sinh\tau}{2m}\frac{1}{(\cosh\tau)^{2m}} \times \left[1 + \sum_{k=1}^{m-1} \frac{\Gamma(m-k)\Gamma(m+1/2)}{\Gamma(m)\Gamma(m-k+1/2)}(\cosh\tau)^{2k}\right] + \frac{(2m-1)!!}{(2m)!!}\arctan(\sinh\tau). \quad (62)$$

In this appendix we show that the two equations written above are special cases of equation (28).

For $A' = 2m \geq 1$ an even integer, the hypergeometric function reduces to a polynomial of order $1 - A'/2$

$$F[-m, b, c; z] = \sum_{k=0}^{m} \frac{(-m)_k (b)_k}{(c)_k}\frac{z^k}{k!}, \quad (63)$$

where

$$(z)_k = \frac{\Gamma(z+k)}{\Gamma(z)}; \qquad (z)_0 = 1. \tag{64}$$

Using equation (63) and the following properties

$$\Gamma(z+1) = z\Gamma(z)$$
$$\Gamma(-z) = \frac{\pi \csc(\pi z)}{-z\Gamma(z)},$$

equation (28) reduces to equation (61).

For $A' = (2m+1)$, we obtain equation (62) from equation (28) by using the properties

$$-i\sinh(\tau)F[1, 1-A'/2, 3/2-A'/2; (\cosh\tau)^2] = F[1/2-A'/2, 1/2, 3/2-A'/2; (\cosh\tau)^2] = F[-m, 1/2, 1-m; (\cosh\tau)^2],$$

and

$$F[-m, 1/2, 1-m; (\cosh\tau)^2] = \frac{1}{2^m}\Gamma(1-m)(-\cosh\tau)^m P_m^m(i\sinh\tau), \tag{65}$$

where

$$P_m^m(z) = (z^2-1)^{m/2}\frac{d^m}{dz^m}P_m(z), \tag{66}$$

with $P_m(z)$ Legendre polinomials.

Highlight Paper

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